Bose-Einstein condensation in variable dimensionality

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We introduce dimensional perturbation techniques to Bose-Einstein condensation of inhomogeneous alkali-metal gases. The perturbation parameter is $\delta=1/\kappa$, where κ depends on the effective dimensionality of the condensate and on the angular momentum quantum number. We derive a simple approximation that is more accurate and flexible than the $N\rightarrow\infty$ Thomas-Fermi ground-state approximation of the Gross-Pitaevskii equation. The approximation presented here is well suited for calculating properties of states in three dimensions and in low-effective dimensionality, such as vortex states in a highly anisotropic trap.

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I. INTRODUCTION

The most commonly used approach to describe a dilute gas of atoms in a Bose-Einstein condensate (BEC) at $T=0$ is mean-field theory, which takes the form of the time-independent Gross-Pitaevskii equation (GPE)

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})+V_{trap}(r)\psi(\mathbf{r})+NU_3|\psi(\mathbf{r})|^2\psi(\mathbf{r})=\mu\psi(\mathbf{r}), \tag{1}$$

where the three-dimensional coupling constant is $U_3=(4\pi\hbar^2a)/m$, a is the s-wave scattering length, V_{trap} is the external trapping potential, μ is the chemical potential, and N is the number of condensate atoms. The complex order parameter $\psi(\mathbf{r})$ is referred to as ''the wave function of the condensate.''

The $N\rightarrow\infty$ Thomas-Fermi approximation (TFA) has been proven to be a highly successful analytical approximation of the GPE [1,2]. The strength of the $N\rightarrow\infty$ TFA is its simplicity: neglecting the kinetic energy results in a simple approximation of the ground-state condensate density that is effective in analyzing properties of large-N condensates. For condensates with a moderate number of atoms and condensates with attractive interactions, the TFA breaks down. The kinetic energy is important in each case, especially in the latter, where the kinetic energy is necessary to prevent collapse. The effects of attractive interactions have been studied using approximation techniques such as variational trial wave functions [3–5]. Other approximations that have been employed to extend the TF regime of validity include the $\hbar\rightarrow0$ TFA [6], a method that uses two-point Padé approximants between the weakly and strongly interacting limits of the ground state [7], and a variational method for anisotropic condensates [8].

With the study of BEC in highly anisotropic traps as a motivation, we use a perturbation formalism that permits the effective dimensionality D to vary. Because it readily allows one to approximate quantities in any dimension, such a formalism is ideally suited for condensates in the anisotropic traps used in many laboratories where the condensate can be effectively one, two, or three dimensional. The perturbation parameter is $\delta=1/\kappa$, where κ depends on the effective dimensionality of the condensate and on the angular momentum quantum number. The $\delta\rightarrow0$ limit becomes an exactly soluble problem, the solution of which is used by the various dimensional-scaling methods as the starting point for the solution of the full three-dimensional problem [9–11]. The $\delta\rightarrow0$ approximation to the condensate density, which retains part of the kinetic energy, is quite accurate for both a large and moderate number of atoms in the BEC ground state, and the dimensional-scaling formalism, which treats the dimensionality as a parameter, is advantageous when studying condensates of low-effective dimensionality due to extreme trap anisotropy. The centrifugal term in the $\delta\rightarrow0$ density also makes it a good physical starting point for treating vortex states.

II. $N\rightarrow\infty$ THOMAS-FERMI APPROXIMATION TO THE GROUND STATE

In the case of positive scattering length, the repulsive interaction causes the density to become flat, and the kinetic energy of the condensate becomes negligible in the $N\rightarrow\infty$ limit. This limit of the GPE results in the highly successful classical approximation for the density of the condensate ground state known as the Thomas-Fermi approximation ($N\rightarrow\infty$ TFA). The $N\rightarrow\infty$ TFA for the ground state in a three-dimensional isotropic trap is [1,2]

$$\rho_{TF}(r)=|\psi|^2=\frac{1}{NU_3}\left(\mu_{TF}-\frac{1}{2}m\omega^2r^2\right) \tag{2}$$

for $\mu_{TF}\geqslant\frac{1}{2}m\omega^2r^2$ and $\rho_{TF}=0$ elsewhere. Equation (2) provides an excellent description of the condensate ground-state density in the bulk interior. This approximation breaks down near the surface of the gas where the density is not flat; the wave function must vanish smoothly, making the kinetic energy appreciable in the boundary layer. The chemical potential is obtained from continuity and normalization of Eq. (2). In oscillator units, one finds

$$\mu_{TF}=\frac{1}{2}R^2, \tag{3}$$

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where $R=(15Na/a_{ho})^{1/5}$ is the Thomas-Fermi classical cutoff radius in oscillator units of length $a_{ho}=\sqrt{\hbar/m\omega}$.

Boundary layer theory techniques have been employed to obtain corrections to the $N\rightarrow\infty$ TFA at the condensate surface where the gradient of the density is no longer small [13–15]. The leading order correction to the ground-state chemical potential due to the boundary layer at the surface is of order $R^{-4}\ln(R)$.

III. EFFECTIVE DIMENSIONALITY

We use a perturbation formalism where the effective dimensionality D of the condensate is allowed to vary. The effective dimensionality of the condensate depends on the relative size of the confinement length in each of the three spatial dimensions. Most experimentally realized traps are axially symmetric, with some having a high degree of anisotropy [16,17]. In the case of axial symmetry, the trapping potential takes the form, $V_{trap}(r)=\frac{1}{2}m\omega_{\perp}^2r_{\perp}^2+\frac{1}{2}m\omega_z^2z^2=\frac{1}{2}m\omega_{\perp}^2(r_{\perp}^2+\lambda^2z^2)$, where $\lambda=\omega_z/\omega_{\perp}$ is a measure of the degree of anisotropy. The system reduces to a three-dimensional isotropic condensate for $\lambda=1$. In the small-(large-) λ limits, the system reduces to an effective one-(two-)dimensional isotropic condensate. It is conceivable that an isotropic Hamiltonian in a fractional-dimensional space could be used to describe experimental condensates for intervening values of λ, but we will focus our attention on integer dimensions.

As an illustration, consider $\lambda\gg1$, where the motion of the atoms in the z direction becomes frozen and their motion is described by a Gaussian of small width. To determine the two-dimensional (2D) effective coupling constant, we assume the wave function in Eq. (1) is separable: $\psi(\mathbf{r})=\psi_2(r_{\perp})\chi(z)$, where $\chi(z)$ is assumed to be a Gaussian, and operating with $\int dz\chi^*$, one finds an effective 2D GPE

$$\left(-\frac{\hbar^2}{2m}\nabla_2^2+\frac{1}{2}m\omega_2^2r^2+N_2U_3|\psi_2|^2\right)\psi_2=\mu_2\psi_2, \quad (4)$$

which has the same form as Eq. (1), but r is the 2D radius, $\omega_2=\omega_{\perp}$, $\mu_2=\mu-\hbar\omega_z/2$, $N_2=N\int dz|\chi|^4$. Requiring that ψ_2 and χ be normalized to unity, $\int dz|\chi|^4$ has units of 1/length; thus, we interpret N_2 as the number of atoms in the 2D condensate per unit length along the z axis. In our subsequent scalings, we will adopt a notation for the number of atoms that is similar to that of Jackson *et al.* [12]. For $\lambda\ll1$, one may assume the motion in the radial direction in the xy plane is described by a Gaussian of small width, $\chi(r_{\perp})$, and, following the same procedure, one obtains a 1D equation analogous to Eq. (4), where N_1 would represent the number of atoms in the 1D condensate per unit area in the xy plane.

IV. GPE IN VARIABLE DIMENSIONALITY

We begin by explicitly generalizing the nonlinear Schrödinger equation (NLSE) Eq. (1), to D dimensions where r becomes the radius of a D-dimensional sphere with D-1 remaining angles. The Laplacian is generalized to D dimensions (Bohn, Esry, and Greene [18] and Watson and McKinney [19] treat the Laplacian in a similar fashion, in which they use hyperspherical coordinates to define a mean condensate radius), and the potential terms retain their three-dimensional form; the coupling constant is generalized in the final scaling. We obtain the Schrödinger equation

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right)+\frac{L_{D-1}^2}{r^2}\right]+\frac{1}{2}m\omega^2r^2+NU_3|\psi(\mathbf{r})|^2\right\}\psi(\mathbf{r})=\mu\psi(\mathbf{r}), \quad (5)$$

where L_{D-1}^2 is a generalized angular momentum operator depending on D-1 angles with eigenvalues $-l(D+l-2)$ [20]; the angular momentum quantum number l is nonnegative. Substituting these eigenvalues and introducing the radial Jacobian factor in a transformation of the wavefunction $\phi(r)=r^{(D-1)/2}\psi(r)$ to eliminate the first derivative terms, we find

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2}-\frac{(D-1)(D-3)}{4r^2}-\frac{l(D+l-2)}{r^2}\right]+\frac{1}{2}m\omega^2r^2+NU_3|\psi(\mathbf{r})|^2\right\}\phi(\mathbf{r})=\mu\phi(\mathbf{r}). \quad (6)$$

Finally, we make two sets of scalings to arrive at the NLSE in dimensionally scaled oscillator units. The first scaling is a purely dimensional scaling: $r=\kappa^2\tilde{r}$, $\tilde{\omega}=\kappa^3\omega$, $\tilde{\mu}=\kappa^2\mu$, and $\tilde{\psi}=\kappa^D\psi$, where $\kappa=D+2l$. The final scaling is to scaled oscillator units (denoted by bars): $\tilde{r}=\tilde{a}_{ho}\bar{r}$, $\tilde{\mu}=\hbar\tilde{\omega}\bar{\mu}$, and $\bar{\psi}=\tilde{a}_{ho}^{D/2}\tilde{\psi}$, where $\tilde{a}_{ho}=\sqrt{\hbar/m\tilde{\omega}}$. Combining these two scalings, we arrive at

$$\left\{-\frac{1}{2}\delta^2\frac{\partial^2}{\partial\bar{r}^2}+\frac{1-4\delta+3\delta^2}{8\bar{r}^2}+\frac{1}{2\bar{r}^2}+\bar{g}_D|\bar{\psi}(\bar{\mathbf{r}})|^2\right\}\bar{\phi}(\bar{\mathbf{r}})=\bar{\mu}\bar{\phi}(\bar{\mathbf{r}}), \quad (7)$$

where everything is now in dimensionally scaled oscillator units and $\delta=1/\kappa$. For the effective dimensions of primary interest in this paper, the dimensionally scaled coupling constants are, for 3D, $\bar{g}_3=g_3/\kappa^{5/2}$, where $g_3=4\pi N_3a/a_{ho}$ and N_3 is the number of condensate atoms; and for 2D, $\bar{g}_2=g_2/\kappa^2$, where $g_2=4\pi N_2a$ and N_2 represents the number of atoms in the 2D condensate per unit length along the z axis. The definition of N_2 makes g_2 dimensionless. For general D,

$$\bar{g}_D=\frac{4\pi N_Da}{\kappa^{(D+2)/2}a_{ho}^{D-2}}, \quad (8)$$

making Eq. (7) valid for describing condensates in any effective dimension. In the next section, we describe a simple and accurate zeroth-order approximation to Eq. (7).

V. ZEROTH-ORDER DENSITY

It has been pointed out by Schuck and Viñas [6] that the true TF limit ($\hbar \to 0$ as originally applied to the case of Fermi statistics [21]) is not equivalent to $N \to \infty$, and they show that the $\hbar \to 0$ TF limit for bosons does not neglect the kinetic energy for the ground state. The $N \to \infty$ TFA to the ground state is too harsh on the kinetic energy for a moderate number of atoms. A less harsh and nearly as simple approximation is the zeroth-order ($\delta \to 0$) approximation of Eq. (7). Unlike the ground-state $N \to \infty$ TFA, which neglects the entire kinetic energy, our zeroth-order approximation of the generalized GPE neglects the derivative part of the kinetic energy but retains a centrifugal term. For vortex states, one understands this term as being a centrifugal barrier due to quantized circulation, which pushes atoms away from the axis of rotation. This centrifugal barrier arises from the condensate phase: $|\nabla S|^2$ ($\psi = \sqrt{\rho} e^{iS}$, where ρ is the condensate density and S is the spatially dependent condensate phase; then the condensate velocity is given by $\mathbf{v} = \hbar/m \nabla S$). For the ground state, this centrifugal term in the zeroth-order density has an alternate, quantum-mechanical interpretation, which helps explain its good agreement with numerical calculations. We discuss this interpretation in Sec. VI.

The $\delta \to 0$ limit of the angular-dimensional perturbation parameter in Eq. (7) results in the following zeroth-order density in scaled oscillator units:

$$\rho(\bar{r}) = |\bar{\psi}|^2 = \frac{1}{\bar{g}_D}\left(\bar{\mu} - \frac{1}{8\bar{r}^2} - \frac{1}{2}\bar{r}^2\right), \tag{9}$$

for $\bar{R}_o(\bar{\mu}) \leqslant \bar{r} \leqslant \bar{R}_{max}(\bar{\mu})$ and $\rho = 0$ elsewhere. The normalization condition becomes

$$\Omega(D) \int_{\bar{R}_o(\bar{\mu})}^{\bar{R}_{max}(\bar{\mu})} d\bar{r}\, \bar{r}^{D-1} |\bar{\psi}|^2 = 1, \tag{10}$$

where $\Omega(D) = 2\pi^{D/2}/\Gamma(D/2)$.[1] The $\delta \to 0$ limit can be thought of as a large-D or large-l limit.

Equation (9) is valid where the density is non-negative. In addition to the $N \to \infty$ TF-like classical cutoff radius near the surface $\bar{R}_{max}$, the centrifugal term requires that another cutoff be defined, $\bar{R}_o$, slightly removed from the origin, to satisfy the requirement that the density be non-negative. In terms of the chemical potential, the cutoff radii in scaled oscillator units are defined as

$$\bar{R}_o^2(\bar{\mu}) = \bar{\mu} - \sqrt{\bar{\mu}^2 - \frac{1}{4}},$$

$$\bar{R}_{max}^2(\bar{\mu}) = \bar{\mu} + \sqrt{\bar{\mu}^2 - \frac{1}{4}}. \tag{11}$$

In regular oscillator units (a_{ho}),

$$R_o^2(\mu) = \mu - \sqrt{\mu^2 - \frac{\kappa^2}{4}},$$

$$R_{max}^2(\mu) = \mu + \sqrt{\mu^2 - \frac{\kappa^2}{4}}. \tag{12}$$

Notice for the ground state in the strongly interacting regime that $\mu \gg 1$ and the cutoff radii for the ground state become $N \to \infty$ TF-like: $R_o \approx 0$ and $\mu \approx R_{max}^2/2$; the strongly interacting limit, or, equivalently, the $N \to \infty$ limit of our zeroth-order approximation collapses to the $N \to \infty$ TFA, as expected. (For finite N, as will be shown later, our zeroth-order approximation gives better agreement with the numerical solution of the GPE than the $N \to \infty$ TFA.) Using the integration limits defined in Eq. (11), along with the condensate density defined in Eq. (9), the normalization condition (Eq. 10) gives an equation for the zeroth-order chemical potential that is easily solved in any dimension. (See Sec. VII, where this procedure is illustrated for two dimensions.)

Once the chemical potential is calculated, it is then used in Eq. (9) to complete the description of the zeroth-order wave function. One can then calculate the energy from

$$E/N = \int d^D\mathbf{r}\left[\frac{\hbar^2}{2m}|\nabla\psi|^2 + \frac{1}{2}m\omega^2 r^2\psi^2 + \frac{g_D}{2}\psi^4\right]$$
$$= E_{kin}/N + E_{ho}/N + E_{int}/N, \tag{13}$$

or in the zeroth-order approximation and scaled oscillator units,

$$\bar{E}/N \approx \Omega(D) \int_{\bar{R}_o(\bar{\mu})}^{\bar{R}_{max}(\bar{\mu})} \bar{r}^{D-1} d\bar{r}\left[\frac{1}{8\bar{r}^2}\bar{\psi}^2 + \frac{1}{2}\bar{r}^2\bar{\psi}^2 + \frac{\bar{g}_D}{2}\bar{\psi}^4\right]$$
$$\approx \bar{E}_{kin}/N + \bar{E}_{ho}/N + \bar{E}_{int}/N. \tag{14}$$

VI. GROUND STATE IN THREE DIMENSIONS

The numerical effect of the centrifugal-like term in the zeroth-order approximation on the ground state of a stationary condensate is clear from Fig. 1, where we compare the numerical solution of the GPE chemical potential with our zeroth-order approximation and the $N \to \infty$ TFA for up to 10 000 ^{87}Rb atoms in a spherical trap. Our zeroth-order approximation is more accurate than the $N \to \infty$ TFA for all N, most notably for a moderate number of atoms. The accuracy of the zeroth-order approximation is comparable to boundary layer corrections: the zeroth-order approximation is slightly more accurate for a smaller coupling constant, while the boundary layer theory is slightly more accurate for a larger

[1] As the GPE is nonlinear, one cannot treat excited-l states (vortices) for $D > 2$ radially symmetric traps in the usual manner of separating the wave function into radial and angular parts. Presently, however, vortices in 3D radially symmetric traps are not realized. Vortex states in a 2D isotropic trap do not pose a problem to theory because the spherical harmonic wave function acts as a phase factor. In 1D, the spherical harmonic wave function is a constant and, since there are no angles, one can think of $l = 0$ and $l = 1$ as even and odd parity states.

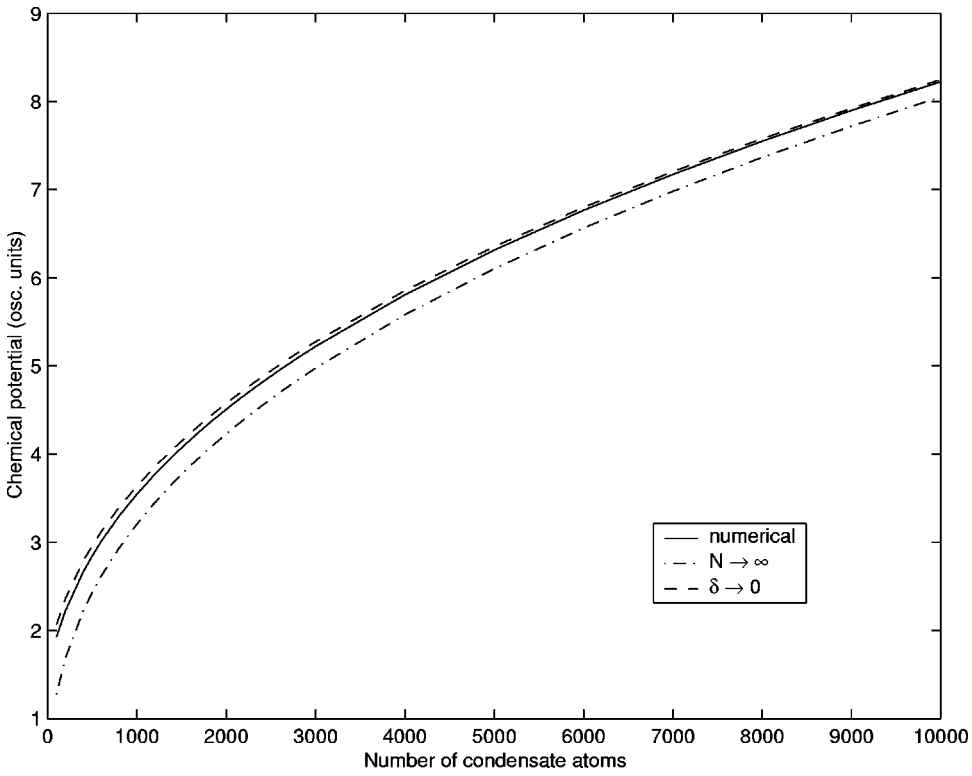

FIG. 1. Chemical potential in oscillator units vs number of condensate atoms for a ^{87}Rb condensate in a 3D isotropic trap, where $a=100$ bohr and $\nu=200$ Hz. The zeroth-order ($\delta\rightarrow 0$) approximation of dimensional perturbation theory presented here (dashed) is in better agreement with the numerical solution of the GPE (solid) than the $N\rightarrow\infty$ Thomas-Fermi approximation (dash dot).

coupling constant, but the difference between all three approximations becomes small for a very large coupling constant.

The correct physical interpretation of this centrifugal-like term, as originally noted by Chatterjee [22], is that it is the component of the kinetic energy needed to satisfy the minimum uncertainty principle. The zeroth-order density includes a centrifugal term from the kinetic energy, which pushes the wave function away from the origin in the ground state as if there were a nonzero quantum of angular momentum; however, the $1/r^2$ contribution to the ground-state density of a nonrotating cloud clearly is not due to any rotational motion of the cloud. This effect, which becomes less pronounced as N increases, is demonstrated in Figs. 2 and 3, which show

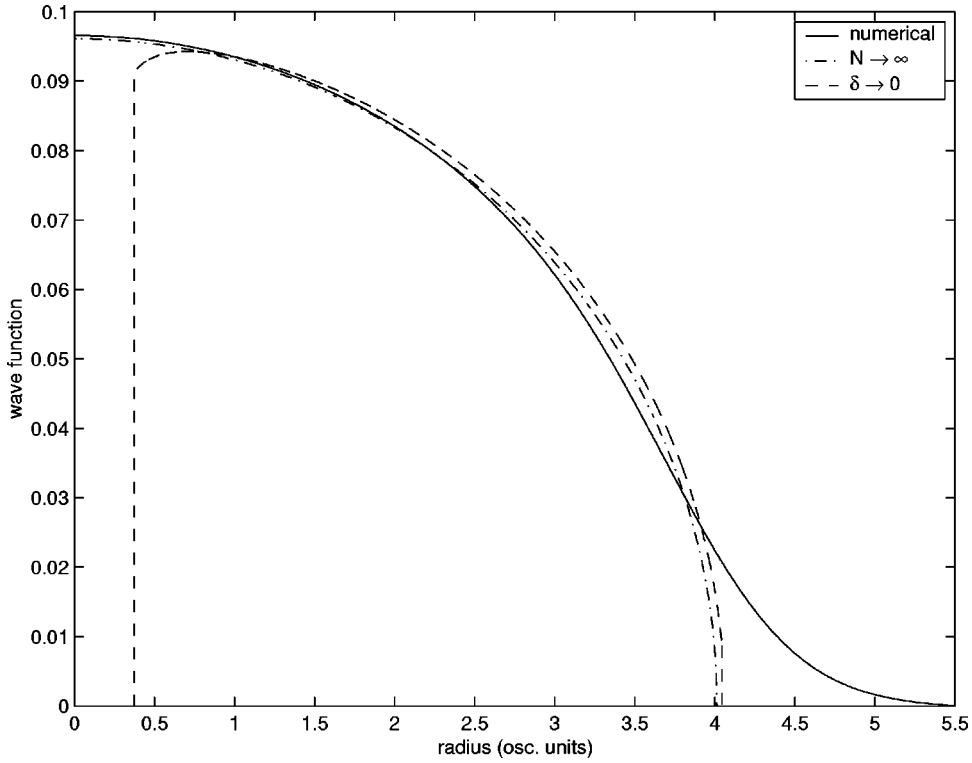

FIG. 2. Ground-state wave functions (ψ, non-Jacobian weighted) for a ^{87}Rb condensate of 10 000 atoms in a 3D isotropic trap, where $a=100$ bohr and $\nu=200$ Hz. These parameters correspond to $g_3\approx 872.04$. Plotted are the numerical solution of the GPE (solid), the $N\rightarrow\infty$ Thomas-Fermi approximation (dash dot) and our zeroth-order ($\delta\rightarrow 0$) approximation (dashed). Our zeroth-order approximation contains an unphysical core near the origin, but the added kinetic energy, which causes the core to appear, is also responsible for the increased accuracy seen in Fig. 1.

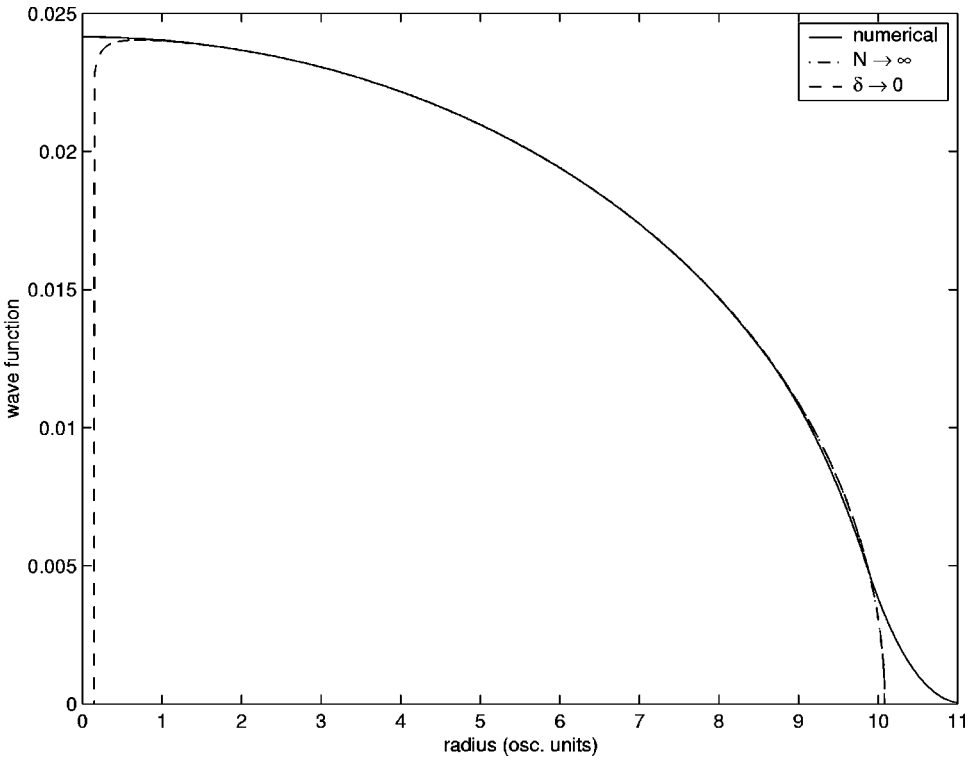

FIG. 3. Same as Fig. 2 but with 10^6 atoms, corresponding to $g_3 \approx 87\,204$. As the number of atoms increases, the unphysical core in our zeroth-order wave function shrinks. Near the origin, the $N \to \infty$ TF and numerically calculated wave functions overlap, while the $N \to \infty$ TF and our zeroth-order wave function overlap in the boundary region. For sufficiently large N, the three wave functions become indistinguishable.

the numerically calculated GPE ground-state non-Jacobian weighted wave function (ψ) along with our zeroth-order approximation and the $N \to \infty$ TFA.

The centrifugal-like term in the lowest order of dimensional-perturbation theory ($\delta \to 0$) can be understood as arising from the requirement that the system's uncertainty product be a minimum [22]. Another way to see how a centrifugal term may arise in the ground state—this time within the $N \to \infty$ TFA—is by applying the Langer modification of WKB theory to the $N \to \infty$ TF density. For vortices, one may not neglect the entire kinetic energy in the $N \to \infty$ limit. A slightly more general $N \to \infty$ TF density than Eq. (2) that includes vortices is

$$\rho_{TF}(r) = |\psi|^2 = \frac{1}{NU_3}\left(\mu_{TF} - \frac{\hbar^2\Lambda^2}{2mr^2} - \frac{1}{2}m\omega^2 r^2\right). \quad (15)$$

For a spherical trap $\Lambda^2 = l(l+1)$, which reduces to the usual ground state $N \to \infty$ TF density for $l=0$, but using the Langer modification, where the correct asymptotic phase of the WKB wave function is obtained by the replacement $l(l+1) \to (l+1/2)^2$ in the centrifugal potential, a centrifugal barrier remains in the ground state

$$\rho_{TF}(r) \to |\psi|^2 = \frac{1}{NU_3}\left(\mu_{TF} - \frac{\hbar^2}{8mr^2} - \frac{1}{2}m\omega^2 r^2\right). \quad (16)$$

The dependence of our perturbation parameter on the angular momentum quantum number suggests that the zeroth-order density will be a good physical starting point for vortex states, which we explore in the next section for $D=2$. The remaining centrifugal term in our zeroth-order approximation is a lowest-order correction to the kinetic energy, which, for a moderate number of atoms, greatly improves the ground-state approximation ($l=0$) over the $N \to \infty$ TFA. For a very large number of atoms, the contribution from the kinetic energy becomes very small, as can be seen by comparing the wave functions in Figs. 2 and 3, for $N = 10^4$ and 10^6 ^{87}Rb atoms, respectively. As the number of atoms increases, our unphysical core becomes smaller than the healing length, eventually vanishing: our zeroth order and the $N \to \infty$ TF wave functions become indistinguishable from the numerical solution for large N.

VII. LOWER DIMENSION

The $\delta \to 0$ density [Eq. (9)] is well suited for describing condensates in the presence of a vortex, where the centrifugal term models the vortex core (see Fig. 4). In this section, we present explicit expressions for the $D=2$ ground state and vortex states. In the angular-dimensional scaling of the GPE, one has considerable freedom in the choice of the scaling parameter $\delta = 1/\kappa$. In the previous section, we used $\kappa = D + 2l$ or, for the ground state, $\kappa = D$. The choice $\kappa = D + 2l - 2$ exactly reduces our expressions below for the chemical potential and energy to a two-dimensional $N \to \infty$ TFA that includes the leading contribution to the kinetic energy due to fluid motion of the condensate [23]. Slightly improved agreement of the zeroth-order energy with the numerical solution of the 2D GPE for a wide range of the mean-field coupling constant can be obtained by choosing $\kappa = D + 2l - 1$, which changes the numerator in the centrifugal term of Eq. (7) to $1 - \delta^2$. Zeroth order predictions show a small amount of variability with the choice of κ, but the results of higher-order perturbation theory should not depend on the particular choice of κ.

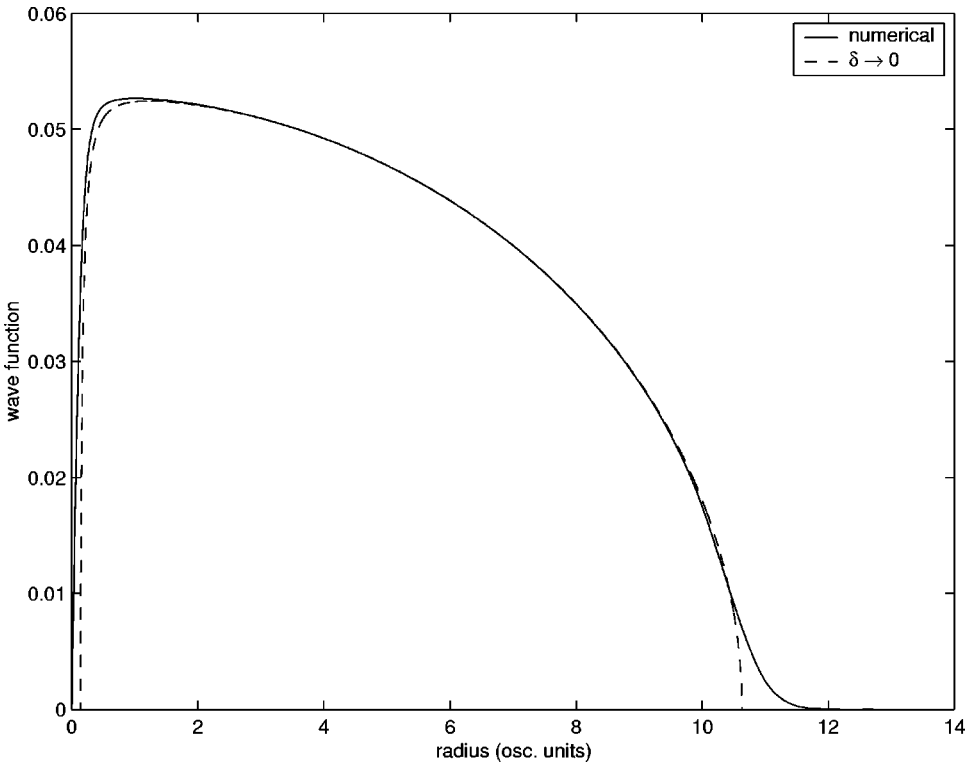

FIG. 4. Comparison of the condensate wave function (ψ, non-Jacobian weighted) with an $l=1$ vortex in a 2D isotropic trap with $g_2=10\,000$. The solid line is the numerical solution of the GPE and the dashed line is our zeroth-order ($\delta\rightarrow 0$) wave function, whose centrifugal term models the vortex core.

Using Eqs. (9), (10), and (11) for $D=2$, one finds that the scaled chemical potential satisfies

$$\frac{\bar{g}_2}{2\pi}=\frac{\bar{\mu}}{2}\sqrt{\bar{\mu}^2-\frac{1}{4}}+\frac{1}{16}\ln\left(\frac{\bar{\mu}-\sqrt{\bar{\mu}^2-\frac{1}{4}}}{\bar{\mu}+\sqrt{\bar{\mu}^2-\frac{1}{4}}}\right). \quad (17)$$

Recalling the earlier conversion relations leading to Eq. (7), the chemical potential, in regular oscillator units, satisfies

$$\frac{g_2}{2\pi}=\frac{\mu}{2}\sqrt{\mu^2-\frac{\kappa^2}{4}}+\frac{\kappa^2}{16}\ln\left(\frac{\mu-\sqrt{\mu^2-\frac{\kappa^2}{4}}}{\mu+\sqrt{\mu^2-\frac{\kappa^2}{4}}}\right). \quad (18)$$

Solving Eq. (17) for the scaled zeroth-order chemical potential $\bar{\mu}$ and using the resulting wave function, Eqs. (9) and (14) give a simple analytical approximation for the 2D energy,

$$E=\frac{2\pi}{3g_2}\left(\mu^2-\frac{\kappa^2}{4}\right)^{3/2}, \quad (19)$$

where we have already converted to regular oscillator units. Equations (18) and (19) for the chemical potential and energy per atom, respectively, are analogous to the results of the $N\rightarrow\infty$ TFA for vortices given in Ref. [23], which includes a kinetic-energy term associated with the fluid motion that is encoded in the wave function's phase. This similarity is due to the zeroth-order $\delta\rightarrow 0$ limit being a large angular momentum limit or, in the language of hydrodynamics, a large quantum of circulation limit. The zeroth-order approximation for $D=2$ results in a shifted TF-like energy spectrum, whose ground-state approximation is, just as for $D=3$, more accurate than the $N\rightarrow\infty$ TFA for any coupling constant, most noticeably for a smaller coupling. For the energy of a single charge vortex located at the center of the trap, the above expressions are more accurate than the unregulated $N\rightarrow\infty$ TFA in Ref. [23] for a moderately sized coupling constant, and slightly less accurate for a very large coupling. For our zeroth-order approximation and $N\rightarrow\infty$ TFA, respectively, the relative errors in the first vortex state energy are 0.56 and $-0.88\,\%$ for $g_2=1000$; and 0.015 and $-0.012\,\%$ for $g_2=100\,000$.

VIII. CONCLUSIONS

We allow the effective dimensionality of the condensate to be a variable quantity, and we use the parameter $\delta=1/\kappa$ to scale the GPE in arbitrary dimension, where κ depends on the effective dimensionality of the condensate and on the angular momentum quantum number. We have shown that our zeroth-order ($\delta\rightarrow 0$) limit of the Gross-Pitaevskii equation provides a less severe approximation of the kinetic energy than the $N\rightarrow\infty$ Thomas-Fermi approximation for the ground state, which neglects the entire kinetic energy. The zeroth-order ($\delta\rightarrow 0$) limit is a simple approximation that, in order to satisfy the minimum uncertainty principle, retains a kinetic-energy contribution, rather than neglecting the entire kinetic energy, making it more accurate and flexible than the ground-state $N\rightarrow\infty$ TFA. As shown in Fig. 1, our zeroth-order approximation is more accurate than the $N\rightarrow\infty$ TFA for the chemical potential. This improvement is caused by the centrifugal-like term, which brings in the kinetic energy

needed to satisfy the minimum uncertainty principle and which adds a needed outward push to the wave function. The accuracy of the zeroth-order approximation is comparable to the lowest-order correction due to the boundary layer at the condensate surface. Improved accuracy for the ground state is most noticeable for a moderate number of atoms, the case in which the kinetic energy is most significant. For a sufficiently large number of atoms with positive scattering length, the kinetic energy becomes small for the ground-state chemical potential, and the three approximations converge to the numerical solution of the GPE.

The core near the origin and the presence of the angular momentum quantum number l in the scaling parameter $\delta = 1/\kappa$ make the zeroth-order ($\delta \to 0$) density an especially good starting point for studying properties of vortices. The ground-state $N \to \infty$ TFA is unable to accommodate such states, but it can be extended to include vortices by introducing the gradient of the phase from the Laplacian [23–26]. We expect higher-order, finite-δ corrections to further refine the shape of our zeroth-order density for the ground state and vortex states.

We have shown that the dimensional-scaling formalism is conducive to analysis of condensates of any dimension. We outlined how simple, yet accurate, approximations can be achieved for any effective D, and we demonstrated the improved numerical results for $D=3$. In addition to 3D BEC, the dimensional-scaling formalism provides a useful analytical tool in the study of BEC in lower-effective dimensionality. In future work, we plan to test the approximation presented here on other observables and states of BEC in D dimensions, and we are extending the methods presented herein to D-dimensional cylindrical coordinates, where the anisotropy parameter is included explicitly for treatment of axially symmetric traps with arbitrary anisotropy.

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